Magnetic moments of mesons

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Abstract

Magnetic moments of charged and neutral mesons are calculated with the use of the relativistic Hamiltonian derived from the path integral form of the $q_1\bar{q}_2$ Green's function. The magnetic moments are shown to be expressed via the average quark energies which are defined by the fundamental quantities: the string tension σ, the current quark masses, and the strong coupling constant α_s. Resulting values for vector, axial, and tensor light and K mesons agree well with all available lattice data.

1 Introduction

Magnetic moments of hadrons are important dynamical characteristics, which can be useful in many areas, e.g for nucleons, and can give new information on the hadron dynamics, being also a serious test of the dynamics put in a model. In this paper we shall exploit the QCD dynamics in the form of the relativistic Hamiltonian (RH) of a meson , derived from the QCD path integral [1], which was already extensively used in the studies of hadron spectra without external fields [2]. The extension of the RH to the case of external magnetic field (MF) has been done recently in Ref. [3], where the meson spectrum as a function of MF was calculated.

Within this method the magnetic moments of baryons have also been studied analytically in Ref. [4] and for the lowest hyperons their values, calculated in the first approximation, agree with experiment within 10% accuracy.

(The same accuracy was achieved for the baryon magnetic moments before, in Ref. [5], where the QCD string dynamics was exploited with approximated wave function).

At the same time for last decades a thorough analysis of the hadron magnetic moments is also being done in the lattice QCD (see [6] for the review).

In this paper we are mainly interested in the magnetic moments of vector, axial, and tensor mesons and to this end, develop further the method of Ref. [4], suggested for baryons. For the sake of generality, here we shall use the RH in MF, both for charged and neutral mesons, from Refs. [3, 7] and in this way define all terms in the expansion of the hadron mass in powers of MF. (This is done in Section 2). We show that the meson magnetic moments are easily calculated through the average quark energies ω_i, which here are defined by the minimal set of the QCD fundamental parameters: the string tension σ, α_s, and the current quark masses, not introducing any fitting parameters.

Having done our analytical calculations, in Section 3 we compare our results for light and K mesons with the lattice calculations of the meson magnetic moments [8], [9] and find a good agreement within the accuracy of the lattice and our calculations. Discussion of the results and prospectives are given in Section 4.

2 The Hamiltonian for a meson in magnetic field

The path-integral Hamiltonian for the $q_1\bar{q}_2$ system in MF was derived in Ref. [3], [7] and has the form,

$$H = H_0 + H_\sigma + W, \tag{1}$$

where

$$H_0 = \sum_{i=1}^{2} \frac{\left(\mathbf{p}^{(i)} - \frac{e_i}{2}(\mathbf{B} \times \mathbf{z}^{(i)})\right)^2 + m_i^2 + \omega_i^2}{2\omega_i}, \tag{2}$$

$$H_\sigma = -\frac{e_1 \boldsymbol{\sigma}_1 \mathbf{B}}{2\omega_1} - \frac{e_2 \boldsymbol{\sigma}_2 \mathbf{B}}{2\omega_2}, \tag{3}$$

In (1) the term W contains the confinement potential V_{conf}, the perturbative gluon-exchange (GE) potential V_{GE}, and spin-dependent interaction V_{SD}, as well as the nonperturbative self-energy term ΔM_{SE} [10],

$$W = V\mathrm{conf} + V_{OGE} + V_{SD} + \Delta M_{SE}. \tag{4}$$

All these terms have been introduced and extensively studied in case without MF in Refs. [2], [10]-[12].

In (2),(3) the following basic elements of the path-integral approach enter: the average energies ω_i, which play actually the role of the einbein parameters [13], being defined from the eigenvalues $M_n(\omega_1, \omega_2)$ of the Hamiltonian H.

$$H\Psi = M_n\Psi, \tag{5}$$

using the stationary point equations:

$$\frac{\partial M_n(\omega_1\omega_2)}{\partial\omega_1}|_{\omega_i=\omega_i^{(0)}} = 0, \quad i = 1, 2. \tag{6}$$

As a result $M_n(\omega_1^{(0)}, \omega_i^{(0)})$ is our prediction for the mass of a given meson.

In the case without MF the eigenvalues M_n were already calculated for all kinds of mesons: light-light [2] , heavy-light [14], heavy quarkonia [15], and in all cases good agreement with experiment was obtained. Note, that $M(\omega_i^{(0)}\omega_i^{(0)})$ are the functions of the current quark masses, the string tension σ, and the strong coupling $\alpha_s(r)$, i.e. do not contain any fitting parameters, in contrast to other relativistic model approaches.

Then we introduce the c.m. and relative coordinates of the $q_1\bar{q}_2$ system,

$$\mathbf{R} = \frac{\omega_1\mathbf{z}^{(1)} + \omega_2\mathbf{z}^{(2)}}{\omega_1 + \omega_2}, \quad \boldsymbol{\eta} = \mathbf{z}^{(1)} - \mathbf{z}^{(2)}, \tag{7}$$

and also make an ansatz for the wave function,

$$\Psi(\boldsymbol{\eta}, \mathbf{R}) = \exp(i\Gamma)\varphi(\boldsymbol{\eta}, \mathbf{R}), \tag{8}$$

where $\Gamma = \mathbf{PR} - \frac{\bar{e}}{2}(\mathbf{B}\times\boldsymbol{\eta})\mathbf{R}$. Then one can get the Hamiltonian H_0', acting on $\varphi(\boldsymbol{\eta}, \mathbf{R})$. If the c.m. motion is chosen to associate with the total charge of the meson, equal $e_1 + e_2$., then one has to put $\bar{e} = \frac{e_1 - e_2}{2}$ and the new Hamiltonian H_0' is obtained in the form,

$$H_0' = \frac{\mathbf{P}^2}{2(\omega_1 + \omega_2)} + \frac{(\omega_1 + \omega_2)\Omega_R^2\mathbf{R}_\perp^2}{2} + \frac{\boldsymbol{\pi}^2}{2\tilde{\omega}} + \frac{\tilde{\omega}\Omega_\eta^2\boldsymbol{\eta}_\perp^2}{2} + X_{LP}\mathbf{B}\mathbf{L}_P +$$

$$+X_{L_\eta}\mathbf{B}\mathbf{L}_\eta+X_1\mathbf{P}(\mathbf{B}\times\boldsymbol{\eta})+X_2(\mathbf{B}\times\mathbf{R})(\mathbf{B}\times\boldsymbol{\eta})+X_3\boldsymbol{\pi}(\mathbf{B}\times\mathbf{R})+\frac{m_1^2+\omega_1^2}{2\omega_1}+\frac{m_2^2+\omega_2^2}{2\omega_2}. \tag{9}$$

Here $\mathbf{L}_\eta=(\boldsymbol{\eta}\times\frac{\partial}{i\partial\boldsymbol{\eta}})$, $\mathbf{L}_P=(\mathbf{R}\times\frac{\partial}{i\partial\mathbf{R}})$, $\tilde{\omega}=\frac{\omega_1\omega_2}{\omega_1+\omega_2}$. All coefficients X_i are given in Appendix 1, while Ω_R,Ω_η are following,

$$\Omega_R^2=B^2\frac{(e_1+e_2)^2}{16\omega_1\omega_2} \tag{10}$$

$$\Omega_\eta^2=\frac{B^2}{2\tilde{\omega}(\omega_1+\omega_2)^2}\left[\frac{(e_1\omega_2+\bar{e}\omega_1)^2}{2\omega_1}+\frac{(e_2\omega_1-\bar{e}\omega_2)^2}{2\omega_2}\right]. \tag{11}$$

Our purpose here is to study the first order corrections $O(eB)$, proportional to spin and the angular momentum $\mathbf{L}_\eta$, to the meson mass $M(\omega_1^{(0)},\omega_2^{(0)})$ in the total expansion of the meson mass $M(B)$ in powers of the MF B (here the relation $\mathbf{L}_P\equiv 0$ is assumed):

$$M(B)=M(0)-\boldsymbol{\mu}_S\mathbf{B}+X_{L_\eta}\mathbf{L}_\eta\mathbf{B}+\sum_{n=1}^{\infty}\kappa_nB^n. \tag{12}$$

Taking into account (9), (10), (11), one arrives at the mass formula for a meson in the form,

$$M(B)=\frac{P_z^2}{2(\omega_1+\omega_2)}+\Omega_R(2n_{R\perp}+1)+$$

$$\left\langle\frac{\tilde{\omega}\Omega_\eta^2\boldsymbol{\eta}_{\eta\perp}^2}{2}\right\rangle+\frac{m_1^2+\omega_2^2}{2\omega_1}+\frac{m_2^2+\omega_2^2}{2\omega_2}-\frac{e_1\boldsymbol{\sigma}_1\mathbf{B}}{2\omega_1}-\frac{e_2\boldsymbol{\sigma}_2\mathbf{B}}{2\omega_2}+X_{L\eta}\mathbf{L}_\eta\mathbf{B}+\langle\Delta M_X\rangle. \tag{13}$$

In (13) the term ΔM_{X_i} implies the sum of all terms with coefficients X_i $(i=1,2,3)$.

Now, expanding (13) in powers of B, one can write ($\mathbf{L}_\eta\equiv\mathbf{l}$)

$$M(B)=M(0)-\boldsymbol{\mu}\mathbf{B}+B\kappa_1+B^2\kappa_2+X_l\mathbf{l}\mathbf{B}+O(B^3), \tag{14}$$

where

$$\boldsymbol{\mu}=\frac{e_1\boldsymbol{\sigma}_1}{2\omega_1^{(0)}(B=0)}+\frac{e_2\boldsymbol{\sigma}_2}{2\omega_2^{(0)}(B=0)}-X_l\mathbf{l}. \tag{15}$$

Here the term $B\kappa_1$ can be obtained, expanding $\omega_i^{(0)}(B)$ in B and keeping the first order term, but all these terms do not contribute to the magnetic

moments. Also in (13) $\langle \Delta M_X \rangle$ contributes only to terms $O(B^2)$, if the vector state considered has no internal angular momenta $\mathbf{L}_P$. In this case all terms in ΔM_X vanish in the first order, $(\Psi \Delta M_X \Psi) = 0$.

3 Results for vector, axial, and tensor mesons

The expression (15) allows to calculate the magnetic moments of mesons with different quantum numbers, both for the angular momentum $l = 0$ and $l \neq 0$. Below we perform calculations for the light and K mesons, while in similar way the magnetic moments of heavy-light and heavy-heavy mesons can be also defined.

a) The case of zero internal angular momentum, $l = 0$

The absolute value μ_S of the magnetic moment $\boldsymbol{\mu} = \mu_S \mathbf{S}$ for the S-wave mesons with $l = 0$ can be obtained, taking $S_z = \frac{1}{2}(\sigma_{1z} + \sigma_{2z}) = +1$, so that one can write

$$\mu_S = \frac{e_1}{2\omega_1^{(0)}(B=0)} + \frac{e_2}{2\omega_2^{(0)}(B=0)}, \tag{16}$$

where $e_1(e_2)$ refers to the charge of quark q_1 (antiquark $\bar{q}_2$). Both $\omega_i^{(0)}(B=0)$ are supposed to be found from the same Hamiltonian (1) with $B = 0$, using as the only input $\sigma, \alpha_s(r)$, and the current quark masses m_i ; their values are discussed in details in Appendix 2.

The values of $\omega_i^{(0)}(B = 0)$ for ρ and K^* were taken from [2], [16] and listed in Tables I, II. Notice that the values of $\omega_i^{(0)}$ given there refer to the case when GE interaction with the standard form of $\alpha_s(r)(n_f = 3)$ is taken into account, while in the absence of GE potential their values for the ground states of light and K mesons would be by $\sim 15\%$ smaller (see (A2.14)).

The most simple case refers to $\rho^\pm$, when neglecting quark masses ($m_u = m_d = 0$), the following values were calculated: $\omega_1^{(0)}(B = 0) = \omega_2^{(0)}(B = 0) \equiv \omega_0 = 0.397$ GeV. Then in nuclear magnetons (n.m.)

$$\mu_S(\rho) \equiv \mu_\rho = \frac{e}{2\omega_0} = \frac{M_P}{\omega_0} \text{ (n.m.)}. \tag{17}$$

In this case the magnetic moment, $\mu_\rho(1S) = 2.37$ n.m., appears to be large (see Table I) and close to that calculated on the lattice in Refs. [8], [9], where $\mu_\rho(1S, lat) \simeq 2.4$ n.m. In the same way for higher radial excitations the

magnetic moments $\mu_\rho(2S)$ and $\mu_\rho(3S)$, given in last column of Table I, are obtained. Note, that in our calculations we have neglected the S-D mixing of excited ρ states, as well as the influence of the spin-spin interaction V_{ss} on the value of ω_0, i.e. V_{ss} is considered as the first order correction to the ρ mass, which is defined via ω_0.

The same procedure is used for the $K^{*\pm}$ mesons, for which important values are given in Table II. For K^* a close agreement with lattice data from Refs. [8], [9] also takes place (see Table III).

Of special interest are the magnetic moment of neutral vector meson K^{*0}. As it follows from (15), if $e_1 = -e_2$, then the magnetic moment is proportional to the difference $(\omega_1^{(0)} - \omega_2^{(0)})$, which in its turn is proportional to $(m_1^2 - m_2^2)$ and vanishes, when the current masses of the strange quark and d-quark are taken to be equal, $\tilde{m}_s = m_d$. For $K^{*0}(d\bar{s})$, using (16) with $e_1 = -\frac{e}{3}, e_2 = +\frac{e}{3}$ and taking $\omega_1 = \omega_n, \omega_2 = \omega_s$ for $m_d = 0, m_s = 0.2$ GeV from Table II, one obtains the magnetic moment of $K^{*0}(d\bar{s})$:

$$\mu(K^{*0}) = -0.0972e(\text{ GeV})^{-1} = -0.183 \text{ n.m.}, \tag{18}$$

which is much smaller than the magnetic moment of K^{*+}. The same result is clearly seen in lattice data [8], where for a neutral meson the linear dependence of its magnetic moment on the squared mass m_π^2, which is proportional to m_q, is observed, thus corresponding to small magnetic moment.

b) **The case of nonzero l**

Firstly one can consider the simple situation, when $\mathbf{S} = 0$, $\mathbf{l} \neq 0$ and in this case clearly $\mathbf{J} = \mathbf{l}$ and $\boldsymbol{\mu} = \mu_l \mathbf{J}$. Hence, one has a simple correspondence:

$$\bar{\mu} = \mu_l \cdot \frac{2M_p}{e} \text{ (n.m.)}, \quad \mu_l \equiv -X_{l\eta} = \frac{e_1\omega_2^2 + e_2\omega_1^2}{2\omega_1\omega_2(\omega_1 + \omega_2)}. \tag{19}$$

The values of $\bar{\mu}$ for $l = 1, 2$ are given in Tables II, III.

In another case, if $\mathbf{S} = 1$, one should define the average value of $\boldsymbol{\mu}$ in the situation, when $\mathbf{J}$ is the only vector of the system, as

$$\bar{\mu} = \langle J, J_z|\mu_z|J, J_z\rangle; \quad J_z = J; \quad \bar{\mu} = \sum_{m_s, m_l} |C^{JJ}_{Sm_S, lm_l}|^2 (\mu_S m_S + \mu_l m_l), \tag{20}$$

where $\boldsymbol{\mu}$ is given by

$$\boldsymbol{\mu} = \mu_S \mathbf{S} + \mu_l \mathbf{l}, \quad \mu_S = \frac{e_1}{2\omega_1} + \frac{e_2}{2\omega_2}; \tag{21}$$

and $\mathbf{J} = \mathbf{l} + \mathbf{S}$. As a result, one obtains the averaged magnetic moment in units $e(\mathrm{GeV}^{-1})$,

$$\bar{\mu} = \frac{\mu_S + \mu_l}{2} (J = l = S = 1), \quad \bar{\mu} = \frac{3\mu_l - \mu_S}{2} \quad (J = 1, l = 2, S = 1), \tag{22}$$

where μ_S, μ_l are given in (21) and (19), respectively.

Then from Eq. (22) one can easily find magnetic moments of all meson states with $l = 1, 2$ and $S = 1$; they are given in Table 1 for light mesons and in Table 2 for strange mesons.

These results refer to the positive (negative) charge mesons, while for neutral light mesons with $\omega_1 = \omega_2$ their magnetic moments are identically zero. For the strange neutral mesons, as mentioned above, their magnetic moments are proportional to $(\omega_1 - \omega_2) \sim m_1^2 - m_2^2$ and therefore they are also very small. While using corresponding values of μ_S, μ_l with $e_1 = -e_2$ from (19), (21), one can find the values of $\bar{\mu}$ for any neutral meson with $S = 1$, using (22). It is important that the Eq.(20) is applicable for the states with all possible values of S, l, and J.

4 Discussion of results and conclusions

The main result of our study is the expression (20) for the magnetic moment of an arbitrary meson, expressed through the factors μ_S, μ_l and hence, through the averaged quark energies ω_1, ω_2. The latter are calculated here within the same relativistic Hamiltonian (see Appendix 2), which contains only the first principle input of QCD: current quark masses, string tension, and α_s in coordinate space. In this way magnetic moments of mesons in the LS scheme, neglecting the $L, L\pm 2$ mixing, were calculated. This mixing can be easily included within our method, provided the mixing amplitudes are known from experiment (e^+e^- cross sections of mesons), or from theoretical models.

Our main results for the light and K mesons are presented in Tables 1,2 and in Tables 3-5 they are compared to other calculations. The most significant our result is for the mesons with the spin $S = 1$ (see Table 3),

where the magnetic moments of different mesons are compared to all existing lattice calculations, since both approaches are of the first principle QCD calculations.

From Table 3 one can see an encouraging agreement within the accuracy of the lattice calculations for all mesons considered. This agreement can be further detailed, using dependence of the averaged energies ω_i on varying current quark masses: it can be translated in the lattice study of the magnetic moment dependence on the pion mass squared – see a quantitative analytic analysis of this dependence in a recent publication [17].

In Tables 4,5 calculated magnetic moments are compared to the existing model calculations: the sum rule approach [18]-[20], the constituent quark model [21], the Dyson-Schwinger approach [22]. One can see a reasonable agreement only for the S-wave mesons, like ρ^+, K^{*+}; however, in other cases a strong disagreement is obtained and this implies that the sum rule method is less reliable for higher excitations of mesons. Another line of possible development is the calculation of magnetic moments for growing MF, in which case for some mesons, the energies ω_i are growing like $\sqrt{eB}$ and therefore their magnetic moment decrease. For other mesons, having different spin projection on the MF, their ω_i decrease [3], [7] and in this case their magnetic moments are growing with the MF.

As it is, our results give an additional support for the relativistic Hamiltonian approach, generalized here to the case of arbitrary strong magnetic and electric fields as in Ref. [3]. We have shown that the varying quark energies ω_i and their final stationary point values $\omega_i^{(0)}$, used in our approach, yield the relevant physical information, which can be applied in different directions: to calculate meson masses with and without magnetic field, the meson magnetic or electric moments of different mesons in strong magnetic field. Our results on magnetic moments can be easily generalized to the case of heavy-light or heavy-heavy mesons.

The same method can be used to study other relativistic systems of two or more constituents: atoms or positronium, different hadrons, and also nuclei in strong MF. The relevant physical situation may exist in astrophysics (magnetars) and in colliding ions, as discussed in [3].

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Table 1: The light meson magnetic moments $\bar{\mu}$ ($\sigma = 0.18$ GeV^2, $m_u = m_d = 0$, $\omega(nl) = \omega_1 = \omega_2)^{a)}$

meson state	$\omega(nl)$	$\bar{\mu}$ (n. m.)
1S	0.397	2.37
2S	0.549	1.71
3S	0.667	1.41
1P	0.489	$a_1^+(^3P_1)$ 1.44; $a_2^+(^3P_2)$ 2.88 $b(^1P_1)$ 0.96
2P	0.616	a_1^+ 1.14; a_2^+ 2.28 $b_1(^1P_1)$ 0.76
1D	0.571	a_3^+ 3.29; $\rho(^3D_1)$ 0.411
2D	0.681	a_3^+ 2.76; $\rho(^3D_1)$ 0.345

$^{a)}$ The parameters of $\alpha_{GE}(r)$ are given in Appendix 2, Eqs. (A.2.8)-(A.2.12).

Table 2: K meson magnetic moments ($m_n = 0, m_s = 0.20$ GeV$, \sigma = 0.18$ GeV^2, $\alpha_{GE}^{a)}$

state	ω_n (GeV)	ω_s (GeV)	$\bar{\mu}$ (n.m.)
1S	0.411	0.467	219
2S	0.559	0612	1.73
3S	0.676	0722	1.36
1P	0.500	0.544	$K^{*0}(^3P_2), \ -0.10$ $K_1^*(^3P_1),\ 1.38$ $K_1^*(^1P_1),\ 0.93$ $K_2^*(^3P_2),\ 2.76$
2P	0.625	0.667	
1D	0.580	0.617	$K^*(^3D_1),\ 0.405$
2D	0.689	0.725	$K^*(^3D_1),\ 0.183$

$^{a)}$ See the footnote to Table 1.

Table 3: Magnetic moments (in n.m.) of the lowest vector and axial mesons in comparison with lattice calculations

meson	$\rho^+(1S)$	K^{*+}	$a_1^+(1^3P_1)$	$K_1^{*+}(^3P_1)$	$\rho_1(^3D_1)$	$K^{*+}(^3D_1)$
m.m. (this paper)	2.37	2.194	1.44	1.38	0.411	0.405
m.m.$^{*)}$ lattice [8]	2.4	2.4	1.5	1.5	0.5	0.5
m.m.$^{*)}$ lattice [9]	2.3	2.1	-	-	-	-

$^{*)}$ The values of lattice magnetic moments are taken from the Figs. [8],[9] and their accuracy is $\gtrsim 10\%$.

Table 4: Magnetic moments of vector and axial mesons (in n.m.) in comparison with other model calculations

ρ^+	K^{*+}	$K^{*(0)}$	a_1^+	source
2.37	2.19	-0.183	1.44	this paper
2.4 ± 0.4	2.0 ± 0.4	0.28 ± 0.04	3.8 ± 0.6	[18]
1.92	-	-	-	[20]
2.14	-	-	-	[21]
2.2	2.08	-0..08	-	[22]

Table 5: Magnetic moments (in n.m.) of the lowest tensor mesons

meson	f_2^t	f_2^0	$a_2^{\pm}$	a_2^0	$K_2^{*+}(1430)$	$K_2^{*0}(1430)$
m.m. (this paper)	2.88	0	2.88	0	2.76	-0.10
m.m. [19]	2.1±0.5	0	1.88± 0.4	0	0.75 ± 0.08	0.076 ± 0.008

Appendix 1

Coefficients $X_i, i = L_\eta, LP, 1, 2, 3$

$$X_{L_\eta} = -\frac{e_1\omega_2^2 + e_2\omega_1^2}{2\omega_1\omega_2(\omega_1+\omega_2)} \tag{A1.1}$$

$$X_{LP} = -\frac{e_1+e_2}{2(\omega_1+\omega_2)} \tag{A1.2}$$

$$X_1 = \frac{e_2\omega_1 - e_1\omega_2 - \bar{e}(\omega_1+\omega_2)}{2(\omega_1+\omega_2)^2} \tag{A1.3}$$

$$X_2 = \frac{-\omega_2(\bar{e}-e_1)(e_1\omega_2+\omega_1\bar{e}) + (\bar{e}+e_2)(e_2\omega_1 - \bar{e}\omega_2)\omega_1}{4\omega_1\omega_2(\omega_1+\omega_2)} \tag{A1.4}$$

$$X_3 = \frac{\bar{e}(\omega_1+\omega_2) - e_1\omega_2 + e_2\omega_1}{2\omega_1\omega_2}, \quad \bar{e} = \frac{e_1-e_2}{2}. \tag{A1.5}$$

Appendix 2

The Hamiltonian without magnetic fields

If there are no MF B, the relativistic Hamiltonian H_0 [1] can be presented as

$$\tilde{H}_0 = \frac{\omega_1}{2} + \frac{m_1^2}{2\omega_1} + \frac{\omega_2}{2} + \frac{m_2^2}{2\omega_2} + \frac{\mathbf{p}^2}{2\tilde{\omega}} + V_{\rm static}(r) \equiv T_R + V_{\rm stat}(r), \tag{A2.1}$$

where by derivation the quark mass cannot be chosen arbitrarily and must be equal to the current mass $\bar{m}_q$ for the u, d, and s quarks. In our calculations $\bar{m}_q = 0$ for the u, d quarks, $m_s \simeq \bar{m}_s(1 \text{ GeV}) = 200$ MeV for the s quark [23]. Here the mass m_s is larger than the conventional $\bar{m}_s(2 \text{ GeV}) = 95 \pm 20$ MeV, taken at the scale $\mu = 2$ GeV, and the reason for that difference originates from the fact that in the static interaction the s-quark current mass $m_s(\mu)$ enters at a smaller scale, $\mu \sim 1$ GeV. It is important that the

current quark masses, used in our relativistic Hamiltonian, allow to avoid such fitting parameters as the constituent quark masses, usually present in other models.

In the static interaction, $V_{\rm stat} = V_{\rm conf} + V_{\rm GE}$, the linear confining potential $V_{\rm conf} = \sigma \cdot r$ is taken here with the string tension $\sigma = 0.18$ GeV^2, which cannot be considered as a fitting parameter, since its value follows from the slope of the Regge trajectories for light mesons [2].

The choice of GE potential is important for low-lying light and K mesons, while its influence is much smaller for high excitations. Here we use the vector strong coupling, denoted as $\alpha_{\rm B}(r)$, which possesses the asymptotic freedom property and freezes at large distances at the value $\alpha_{\rm crit}$, and its parameters are not arbitrary, as we show below.

The variables ω_i, entering RH $\tilde{H}_0$, have to be determined from the extremum conditions: $\frac{\partial H_0}{\partial \omega_i} = 0$ $(i = 1, 2)$, that gives

$$\omega_i(nl) = \langle \sqrt{\mathbf{p}^2 + m_i^2} \rangle_{nl} \quad (i = 1, 2). \tag{A2.2}$$

These average energies, $\omega_1(nl)$ and $\omega_2(nl)$, refer to the quark q_1 and the antiquark $\bar{q}_2$, while $\tilde{\omega}$ is the reduced mass, $\tilde{\omega} = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}$, and $\mathbf{l} = \mathbf{l}_1 + \mathbf{l}_2$. Then putting ω_i into Eq. (A2.1), one arrives at a different form of the kinetic energy term, denoted as $T_{\rm R}$:

$$T_R = \sqrt{\mathbf{p}^2 + m_q^2} + \sqrt{\mathbf{p}^2 + m_c^2}. \tag{A2.3}$$

Rigorously, the expression (A2.3) for $T_{\rm R}$ is valid only for $l = 0$, while in general case, for $l \neq 0$, $T = T_{\rm R} + T_{\rm str}$ contains additional kinetic energy term, $T_{\rm str}$, which appears because, besides a standard rotation of a quark and an antiquark, the string rotates itself. It was shown in Refs. [2], [24] that for $l \leq 4$ the contribution from $T_{\rm str}$ is relatively small compared to the e.v. $M_0(nl)$ and therefore $T_{\rm str}$ can be considered as a perturbation. Still its matrix element (m.e.) $\Delta_{\rm str}(nL) = \langle T_{\rm str} \rangle_{nl}$ has to be included in the mass formula of a meson.

Then the e.v. $M_0(nl)$ and the meson w.f. are defined by the spinless Salpeter equation (SSE):

$$[T_{\rm R} + V_{\rm B}(r)]\, \varphi_{nl} = M_0(nl) \varphi_{nl}. \tag{A2.4}$$

However, the spin-averaged meson mass $M(nl) \equiv M_{\rm cog}(nl)$ includes not only the e.v. $M_0(nl)$ (A2.4), but also two additional negative contributions: the

string correction $\Delta_{\rm str}(nl) = \langle H_{\rm str}\rangle_{nl}$ [2], if $l \neq 0$, and the nonperturbative self-energy (SE) term $\Delta_{\rm SE}(nl)$ [10]:

$$M(nl) = M_0(nl) + \Delta_{\rm str}(nl) + \Delta_{\rm SE}(nl). \tag{A2.5}$$

For given quantum numbers n, l the string correction increases for larger l, while for a fixed l it decreases for higher radial excitations. For the $1P$, $1D$ light mesons their values are typically equal to ~ -40 MeV, -70 MeV, respectively (they were calculated using analytical expressions for $\Delta_{\rm str}$ from [1],[2], [24]).

The nonperturbative SE correction to the quark (antiquark) mass is of great importance to provide linear behavior of the Regge trajectories [2]. As shown in Ref. [10], this correction is flavor-dependent, depends on the averaged energy of a quark, being very small for a heavy quark and large for a light (strange) quark:

$$\Delta_{\rm SE} = -\frac{3\sigma}{2\pi}\left(\frac{\eta_1}{\omega_1(nl)} - \frac{\eta_2}{\omega_2(nl)}\right). \tag{A2.6}$$

The factor η_f ($f = 1, 2$) depends on the quark flavor and the vacuum correlation length: $\eta_n = 1.0$ for a light quark, $\eta_s = 0.70$ for the s quark. Notice that the number $3/2$ enters the SE term (A2.6), instead of the number 2 derived before in [10]; this change comes from more exact definition of the vacuum correlation length [25].

From Eq. (A2.6) one can see that the averaged quark energies ω_i play a special role: they determine both the string and the SE contributions, and also enter all spin-dependent potentials. In some potential models a negative overall constant is often introduced in the mass term (which plays the role of a self-energy correction), however, such a constant violates the linear behavior of the Regge trajectories.

The "linear+GE" potential $V_{\rm B}(r)$, was already tested in a large number of previous studies of heavy-light mesons [14] and heavy-quarkonia [15]:

$$V_{\rm B}(r) = \sigma r - \frac{4\alpha_{\rm B}(r)}{3r}, \tag{A2.7}$$

where the vector coupling $\alpha_{\rm B}(r)$ is taken as in background perturbation theory [26] with the freezing value $\alpha_{\rm crit} = 0.495$ ($n_f = 3$).

The vector coupling in coordinate space is defined through the vector coupling $\alpha_B(q^2)$ in the momentum space:

$$\alpha_B(r) = \frac{2}{\pi}\int_0^\infty dq \frac{\sin(qr)}{q}\,\alpha_B(q), \tag{A2.8}$$

which is taken in two-loop approximation,

$$\alpha_B(q) = \frac{4\pi}{\beta_0 t_B}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\ln t_B}{t_B}\right). \tag{A2.9}$$

Here the logarithm,

$$t_B = \frac{q^2 + M_B^2}{\Lambda_B^2}, \tag{A2.10}$$

contains the QCD constant $\Lambda_B(n_f)$, which is defined via the QCD constant $\Lambda_{\overline{MS}}(n_f)$ in the $\overline{MS}$ scheme. The relation between them has been established in [27]:

$$\Lambda_B(n_f) = \Lambda_{\overline{MS}} \exp\left(-\frac{a_1}{2\beta_0}\right), \tag{A2.11}$$

with $\beta_0 = 11 - \frac{2}{3}n_f$ and $a_1 = \frac{31}{3} - \frac{10}{9}n_f$. From the relation (A2.11) one can see that for a given n_f the constant $\Lambda_B(n_f)$ is significantly larger than $\Lambda_{\overline{MS}}$:

$$\Lambda_B^{(5)} = 1.3656\Lambda_{\overline{MS}}^{(5)} \quad (n_f = 5); \Lambda_B^{(3)} = 1.4753\Lambda_{\overline{MS}}^{(3)} \quad (n_f = 3). \tag{A2.12}$$

At present only the QCD constant $\Lambda_{\overline{MS}}^{(5)}$ (for $n_f = 5$) is well known from experimental value of $\alpha_s(M_Z) = 0.1182 \pm 0.0012$; then in two-loop approximation it gives $\Lambda_{\overline{MS}}^{(5)}$(two-loop) = 232(12) MeV. For $n_f = 3$ the QCD constant $\Lambda_{\overline{MS}}$ is not known with a good accuracy and to define it we fix here the freezing value α_{crit}: $\alpha_{\rm crit}(n_f = 3) \simeq 0.495$. In (A2.10) the background mass M_B also enters; its value is proportional to $\sqrt{\sigma}$ and for $\sigma = 0.18$ GeV^2 the number $M_B = 1.0 \pm 0.05$ GeV was extracted from a detailed comparison of the static force in the field correlator method used and in the lattice QCD [28] (here we take $M_B = 1.0$ GeV).

The important feature of the critical couplings is that in the momentum and coordinate space they coincide, $\alpha_B(crit) = \alpha_B(q^2 = 0) = \alpha_B(r \to \infty)$:

$$\alpha_B(crit) = \alpha_B(r \to \infty) = \alpha_B(q = 0) = \frac{4\pi}{\beta_0 t_0}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\ln t_0}{t_0}\right), \tag{A2.13}$$

with $t_0 = t_B(q^2=0) = ln\left(\frac{M_B^2}{\Lambda_B^2}\right)$. Thus in our calculations $\Lambda_B(n_f=3) = 360$ MeV, $M_B = 1.0$ GeV, $\alpha_{crit} = 0.4945$, and $\sigma = 0.18$ GeV^2.

In [2], [16] it was shown that the GE interaction remains important for the ground states of light, K, and ϕ mesons, e.g. if GE interaction is neglected, then

$$\omega_1^{(0)}(1S) = \omega_2^{(0)}(1S) = 335 \text{ MeV for light mesons} \tag{A2.14}$$

$$\left.\begin{array}{l}\omega_1^{(0)}(1S) = 347 \text{ Mev} \\ \omega_2^{(0)}(1S) = 411\end{array}\right\} \text{ for K mesons,}$$

while their values $\omega_1(nl), \omega_2(nl)$ increase if GE interaction is taken into account (see Tables I,II). Thus for a light meson $\omega(1S)$ appears to be $\sim 18\%$ larger and such the growth of $\omega_i(nl)$ is important for more precise definition of the magnetic moments of the ρ and K^* mesons. Notice, that for higher excitations the GE interaction provides an increase of $\omega_i(nl)$ by only $\sim 5\%$.

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